

Rounding of first-order phase transitions and optimal cooperation in scale-free networks

M. Karsai,^{1,2} J-Ch. Anglès d'Auriac,² and F. Iglói^{3,1}

¹*Institute of Theoretical Physics, Szeged University, H-6720 Szeged, Hungary*

²*Centre de Recherches sur les Très Basses Températures, B. P. 166, F-38042 Grenoble, France*

³*Research Institute for Solid State Physics and Optics, H-1525 Budapest, P.O. Box 49, Hungary*

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We consider the ferromagnetic large- q state Potts model in complex evolving networks, which is equivalent to an optimal cooperation problem, in which the agents try to optimize the total sum of pair cooperation benefits and the supports of independent projects. The agents are found to be typically of two kinds: A fraction of m (being the magnetization of the Potts model) belongs to a large cooperating cluster, whereas the others are isolated one man's projects. It is shown rigorously that the homogeneous model has a strongly first-order phase transition, which turns to second-order for random interactions (benefits), the properties of which are studied numerically on the Barabási-Albert network. The distribution of finite-size transition points is characterized by a shift exponent, $1/\bar{\nu}'=0.26(1)$, and by a different width exponent, $1/\nu'=0.18(1)$, whereas the magnetization at the transition point scales with the size of the network, N , as $m \sim N^{-x}$, with $x=0.66(1)$.

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I. INTRODUCTION

Complex networks have been used to describe the structure and topology of a large class of systems in different fields of science, technics, transport, social and political life, etc.; see Refs. [1–4] for recent reviews. A complex network is represented by a graph [5], in which the nodes stand for the agents and the edges denote the possible interactions. Realistic networks generally have three basic properties. The average distance between the nodes is small, which is called the small-world effect [6]. There is a tendency of clustering and the degree distribution of the edges, $P_D(k)$, has a power-law tail, $P_D(k) \approx Ak^{-\gamma}$, $k \gg 1$. Thus the edge distribution is scale free [7], which is usually attributed to growth and preferential attachment during the evolution of the network.

In reality there is some sort of interaction between the agents of a network which leads to some kind of cooperative behavior in macroscopic scales. One thoroughly studied question in this field is the spread of infections and epidemics in networks [8–10]; this problem is closely related to other nonequilibrium processes, such as percolation [11], diffusion [12], the contact process [13] or the zero-range process [14], etc. In another investigation one considers simple magnetic models [15–19], in which the agents are represented by classical (Ising or Potts) spin variables; the interactions are described by ferromagnetic couplings, whereas the temperature plays the role of a disordering field.

In theoretical investigations of the cooperative behavior one usually resorts to some kind of approximations. For example, the sites of the networks are often considered uncorrelated, which is generally not true for evolving networks, such as the Barabási-Albert (BA) network. However, this effect is expected to be irrelevant, as far as the singularities in the system are considered. Also the simple mean-field approach could lead to exact results due to long-range interactions in the networks, which has been checked by numerical simulations [15] and by other, more accurate theoretical methods [19] (Bethe-lattice approach, replica method, etc.). In these calculations the critical behavior of the network is

found to depend on the value of the degree exponent, γ . For sufficiently weakly connected networks with $\gamma > \gamma_u$ ($\gamma_u=5$ for the Ising model) there are conventional mean-field singularities. In the intermediate or unconventional mean-field regime, for $\gamma_u > \gamma > \gamma_c$, the critical exponents are γ dependent. Finally, for $\gamma \leq \gamma_c$, when the average of k^2 , defined by $\langle k^2 \rangle = \int P_D(k) k^2 dk$, as well as the strength of the average interaction, becomes divergent the scale-free network remains in the ordered state at any finite temperature. Since $\gamma_c=3$, in realistic networks with homogeneous interactions this type of phenomena is always expected to occur. In weighted networks, however, in which the strength of the interaction is appropriately rescaled with the degrees of the connected vertices, γ_c is shifted to larger values and therefore the complete phase-transition scenario can be tested [13,19]. We note that the properties of the phase transitions are generally different for undirected (as we consider here) and directed networks [20].

In several models the phase transition in regular lattices is of first order, such as for the q -state Potts model for sufficiently large values of q . Putting these models on a complex network the inhomogeneities of the lattice play the role of some kind of disorder and it is expected that the value of the latent heat is reduced or even the transition is smoothed to a continuous one. This type of scenario is indeed found in a mean-field treatment [17], in which the transition is of first order for $\gamma > \gamma(q)$ and becomes continuous for $\gamma_c < \gamma < \gamma(q)$, where $3 < \gamma(q) < 4$. On the other hand, in an effective medium Bethe lattice approach, one has obtained $\gamma(q) = 3$; thus the unconventional mean-field regime is absent in this treatment [18].

The interactions considered so far were homogeneous, however, in realistic situations the disorder is inevitable, which has a strong influence on the properties of the phase transition. In regular lattices and for a second-order transition Harris-type relevance-irrelevance criterion can be used to decide about the stability of the pure system's fixed point in the presence of weak disorder. On the contrary, for a first-order transition such type of criterion does not exist. In this case

rigorous results assert that in two dimensions (2D) for any type of continuous disorder the originally first order transition softens into a second order one [21]. In three dimensions there are numerical investigations which have shown [22–26] that this kind of softening takes place only for sufficiently strong disorder.

In this paper we consider interacting models with random interactions on complex networks and in this way we study the combined effect of network topology and bond disorder. The particular model we consider is the random bond ferromagnetic Potts model (RBPM) for large values of q . This model besides its relevance in ordering-disordering phenomena and phase transitions has an exact relation with an optimal cooperation problem [27]. This mapping is based on the observation that in the large- q limit the thermodynamic properties of the system are dominated by one single diagram [28] of the high-temperature expansion [29] and its calculation is equivalent to the solution of an optimization problem. This optimization problem can be interpreted in terms of cooperating agents which try to maximize the total sum of benefits received for pair cooperations plus a unit support which is paid for each independent project. For a given realization of the interactions the optimal state is calculated exactly by a combinatorial optimization algorithm which works in strongly polynomial time [27]. The optimal graph of this problem consists of connected components (representing sets of cooperating agents) and isolated sites and its temperature (support) dependent topology contains information about the collective behavior of the agents. In the thermodynamic limit one expects to have a sharp phase transition in the system, which separates the ordered (cooperating) phase with a giant cluster from a disordered (noncooperating) phase, having only clusters of finite extent.

The structure of the paper is the following. The model and the optimization method used in the study for large q is presented in Sec. II. The solution for homogeneous nonrandom evolving networks can be found in Sec. III, whereas numerical study of the random model on the Barabási-Albert network is presented in Sec. IV. Our results are discussed in Sec. V.

II. MODEL AND ITS RELATION WITH OPTIMAL COOPERATION

The q -state Potts model [30] is defined by the Hamiltonian

$$\mathcal{H} = - \sum_{\langle i,j \rangle} J_{ij} \delta(\sigma_i, \sigma_j) \quad (1)$$

in terms of the Potts-spin variables, $\sigma_i = 0, 1, \dots, q-1$. Here i and j are sites of a lattice, which is represented by a complex network in our case and the summation runs over nearest neighbors, i.e., pairs of connected sites.

The couplings, $J_{ij} > 0$, are ferromagnetic and they are either identical, $J_{ij} = J$, which is the case of homogeneous networks, or they are identically and independently distributed random variables. In this paper, we use a quasicontinuous distribution

$$P(J_{ij}) = \frac{1}{l} \sum_{i=1}^l \delta \left[J \left(1 + \Delta \frac{2i-l-1}{2l} \right) - J_{ij} \right], \quad (2)$$

which consists of a large number l of equally spaced discrete values within the range $J(1 \pm \Delta/2)$ and $0 \leq \Delta \leq 2$ measures the strength of disorder.

For a given set of couplings the partition function of the system is convenient to write in the random cluster representation [29] as

$$Z = \sum_G q^{c(G)} \prod_{ij \in G} [q^{\beta J_{ij}} - 1], \quad (3)$$

where the sum runs over all subset of bonds G and $c(G)$ stands for the number of connected components of G . In Eq. (3) we use the reduced temperature $T \rightarrow T \ln q$ and its inverse $\beta \rightarrow \beta / \ln q$, which are of $O(1)$ even in the large- q limit [31]. In this limit we have $q^{\beta J_{ij}} \gg 1$ and the partition function can be written as

$$Z = \sum_{G \subseteq E} q^{\phi(G)}, \quad \phi(G) = c(G) + \beta \sum_{ij \in G} J_{ij}, \quad (4)$$

which is dominated by the largest term, $\phi^* = \max_G \phi(G)$. Note that this graph, which is called the optimal set, generally depends on the temperature. The free-energy per site is proportional to ϕ^* and given by $-\beta f = \phi^* / N$ where N stands for the number of sites of the lattice.

As already mentioned in the Introduction, the optimization in Eq. (4) can be interpreted as an optimal cooperation problem [27] in which the agents, which cooperate with each other in some projects, form connected components. Each cooperating pair receives a benefit represented by the weight of the connecting edge (which is proportional to the inverse temperature) and also there is a unit support to each component, i.e., for each project. Thus by uniting two projects the support will be reduced but at the same time the edge benefits will be enhanced. Finally, one is interested in the optimal form of cooperation when the total value of the project grants is maximal.

From a mathematical point of view, the cost function in Eq. (4), $-\phi(G)$, is submodular [32] and there is an efficient combinatorial optimization algorithm which calculates the optimal set (i.e., set of bonds which minimizes the cost function) exactly at any temperature in strongly polynomial time [27]. In the algorithm the optimal set is calculated iteratively and at each step one new vertex of the lattice is taken into account. Having the optimal set at a given step, say with n vertices, its connected components have the property to contain all the edges between their sites. Due to the submodularity of $-\phi(G)$ each connected component is contracted into a new vertex with effective weights being the sum of individual weights in the original representation. Now adding a new vertex one should solve the optimization problem in terms of the effective vertices, which needs the application of a standard maximum flow algorithm, since any contractions should include the new vertex. After making the possible new contractions one repeats the previous steps until all the vertices are taken into account and the optimal set of the problem is found.

This method has already been applied for 2D [31,33] and 3D [25,26] regular lattices with short range random interactions. As a general result, the optimal graph at low temperatures is compact and the largest connected subgraph contains a finite fraction of the sites, $m(T)$, which is identified by the order parameter of the system. In the other limit, for high temperature, most of the sites in the optimal set are isolated and the connected clusters have a finite extent, the typical size of which is used to define the correlation length, ξ . Between the two phases there is a sharp phase transition in the thermodynamic limit, the order of which depends on the dimension of the lattice and the strength of disorder, Δ .

In the following, the optimization problem is solved exactly for homogeneous evolving networks in Sec. III and studied numerically in random Barabási-Albert networks in Sec. IV.

III. EXACT SOLUTION FOR HOMOGENEOUS EVOLVING NETWORKS

In regular d -dimensional lattices the solution of the optimization problem in Eq. (4) is simple, since there are only two distinct optimal sets, which correspond to the $T=0$ and $T \rightarrow \infty$ solutions, respectively. For $T < T_c^{(0)}$ it is the fully connected diagram, E , with a free energy $-\beta N f = 1 + N \beta J d$ and for $T > T_c^{(0)}$ it is the empty diagram, \emptyset , with $-\beta N f = N$. [The superscript in $T_c^{(0)}$ is related to the nonrandom system, i.e., when the strength of disorder in Eq. (2) is $\Delta=0$.] In the proof we make use of the fact that any edge of a regular lattice, e_1 , can be transformed to any another edge, e_2 , through operations of the automorphy group of the lattice. Thus if e_1 belongs to some optimal set, then e_2 belongs to an optimal set, too. Furthermore, due to submodularity the union of optimal sets is also an optimal set, from which follows that at any temperature the optimal set is either \emptyset or E . By equating the free energies in the two phases we obtain, for the position of the transition point, $T_c^{(0)} = Jd/(1-1/N)$, whereas the latent heat is maximal: $\Delta e/T_c^{(0)} = 1-1/N$.

In the following, we consider the optimization problem in homogeneous evolving networks which are generated by the following rules: We start with a complete graph with 2μ vertices; at each time step we add a new vertex; which is connected to μ existing vertices.

In definition of these networks there is no restriction to which way the μ existing vertices are selected. These could be chosen randomly, as in the Erdős-Rényi model [34], or one can follow some defined rule, like the preferential attachment in the BA network [7]. In the following we show that for such networks the phase-transition point is located at $T_c^{(0)} = J\mu$ and for $T < T_c^{(0)}$ ($T > T_c^{(0)}$) the optimal set is the fully connected diagram (empty diagram), as for the regular lattices. Furthermore, the latent heat is maximal: $\Delta e/T_c^{(0)} = 1$.

In the proof we follow the optimal cooperation algorithm [27] outlined in Sec. II, and in application of the algorithm we add the vertices one by one in the same order as in the construction of the network. First we note that the statement is true for the initial graph, which is a complete graph; thus the optimal set can be either fully connected, having a free-

energy $-\beta 2\mu f = 1 + \mu(2\mu-1)\beta J$, or empty, having a free energy $-\beta 2\mu f = 2\mu$. Thus the transition point is indeed at $T = T_c^{(0)}$. We suppose then that the property is satisfied after n steps and add a new vertex, v_0 . Here we investigate the two cases, $T \leq T_c^{(0)}$ and $T \geq T_c^{(0)}$, separately.

If $T \leq T_c^{(0)}$, then according to our statement all vertices of the original graph are contracted into a single vertex, s , which has an effective weight, $\mu \times J/T > \mu J/T_c^{(0)} = 1$, to the new vertex, v_0 . Consequently in the optimal set s and v_0 are connected, in accordance with our statement.

If $T \geq T_c^{(0)}$, then all vertices of the original graph are disconnected, which means that for any subset, S , having $n_s \leq n$ vertices and e_s edges, one has $n_s \geq e_s J T + 1$. Let us denote by $\mu_s \leq \mu$ the number of edges between v_0 and the vertices of S . One has $\mu_s J/T \leq \mu J/T \leq \mu J/T_c^{(0)} = 1$, so that for the composite $S+v_0$ we have $n_s+1 \geq e_s J T + 1 + \mu_s J/T$, which proves that the vertex v_0 will not be connected to any subset S and thus will not be contracted to any vertex.

This result, i.e., a maximally first-order transition of the large- q state Potts model holds for a wide class of evolving networks which satisfy the construction rules presented above. This is true, among others, for randomly selected sites, for the BA evolving network which has a degree exponent $\gamma=3$, and for several generalizations of the BA network [1] including nonlinear preferential attachment, initial attractiveness, etc. In these latter network models the degree exponent can vary in a range of $2 < \gamma < \infty$. It is interesting to note that for uncorrelated random networks with a given degree distribution the q -state Potts model is in the ordered phase [17,18] for any $\gamma \leq 3$. This is in contrast to evolving networks in which correlations in the network sites result in the existence of a disordered phase for $T > T_c^{(0)}$, at least for large q .

IV. NUMERICAL STUDY OF RANDOM BARABÁSI-ALBERT NETWORKS

In this section, we study the large- q state Potts model in the BA network with a given value of the connectivity, $\mu = 2$, and the size of the network varies between $N=2^6$ to $N=2^{12}$. The interactions are independent random variables taken from the quasicontinuous distribution in Eq. (2) having $l=1024$ discrete peaks and we fix $J=1$. The advantage of using quasicontinuous distributions is that in this way we avoid extra, nonphysical singularities, which could appear for discrete (e.g., bimodal) distributions [31]. For a given size we have generated 100 independent networks and for each we have 100 independent realizations of the disordered couplings.

A. Magnetization and structure of the optimal set

In Fig. 1 the temperature dependence of the average magnetization is shown for various strength of disorder, Δ , for a BA network of $N=1024$ sites. It is seen that the sharp first-order phase transition of the pure system with $\Delta=0$ is rounded and the magnetization has considerable variation within a temperature range of $\sim \Delta$. The phase transition seems to be continuous even for weak disorder. Close to the

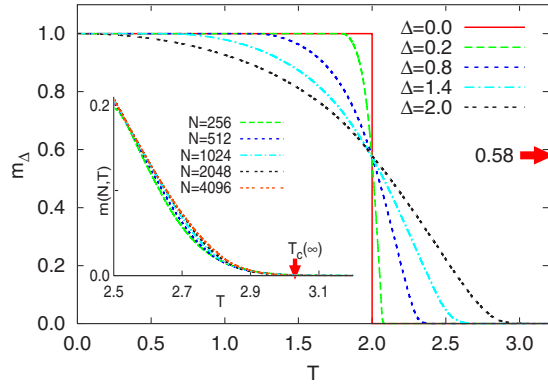


FIG. 1. (Color online) Temperature dependence of the average magnetization in a BA network of $N=1024$ sites for different strengths of the disorder, Δ . At $T=T_c^{(0)}=2$ the magnetization is independent of $\Delta>0$ and its value is indicated by an arrow. Inset: The average magnetization for uniform disorder, $\Delta=2$, close to the transition point for different finite sizes. The arrow indicates the critical point of the infinite system.

transition point the magnetization curves for uniform disorder ($\Delta=2$) are presented in the inset of Fig. 1, which are calculated for different finite systems.

Some features of the magnetization curves and the properties of the phase transition can be understood by analyzing the structure of the optimal set. For low enough temperature this optimal set is fully connected, i.e., the magnetization is $m=1$, which happens for $T<T_c^{(0)}-\Delta$. Indeed, the first sites with $k=\mu=2$ (i.e., those which have only outgoing edges) are removed from the fully connected diagram, if the sum of the connected bonds is $\sum_{i=1}^{\mu} J_i < T$, which happens within the temperature range indicated above. From a similar analysis follows that the optimal set is empty for any finite system for $T>T_c^{(0)}+\Delta$. The magnetization can be estimated for $t=T-(T_c^{(0)}-\Delta)\ll 1$ and the correction is given by $1-m\sim t^\mu$. For the numerically studied model with $\mu=2$ and $\Delta=2$, we have $m(T)\approx 1-T^2/8$, which is indeed a good approximation for $T<1$. In the temperature range $T_c^{(0)}-\Delta<T<T_c^{(0)}+\Delta$ typically the sites are either isolated or belong to the largest cluster. There are also some clusters with an intermediate size, which are dominantly two-site clusters for $T<T_c^{(0)}$ and their fraction is less than 1%, as shown in Fig. 2. The fraction of two-site clusters for $\Delta=2$ and $T<T_c^{(0)}=2$ can be estimated as follows. First, we note that since they are not a part of the biggest cluster they can be taken out of a fraction of $p_1=1-m(T)$ sites. Before being disconnected a two-site cluster has typically three bonds to the biggest cluster, denoted by J_1, J_2 , and J_3 . When it becomes disconnected we have $J_1+J_2+J_3<T$, which happens with a probability $p_2=T^3/48$. At the same time the coupling within the two-site cluster should be $J_4>T$, which happens with probability $p_3=(2-T)/2$. Thus the fraction of two-site clusters is approximately $n_2\approx p_1\times p_2\times p_3\approx T^5(2-T)/768$, which describes well the general behavior of the distribution in Fig. 2.

In the temperature range $T>T_c^{(0)}$ the intermediate clusters have at least three sites and their fraction is negligible, which is seen in Fig. 2. Consequently, the intermediate size clusters do not influence the properties of the phase transition in the

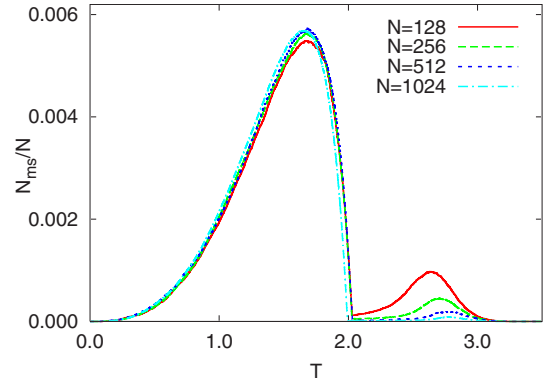


FIG. 2. (Color online) Fraction of intermediate size clusters as a function of the temperature.

system. In the ordered phase, $T<T_c$, the largest connected cluster contains a finite fraction of $m(T)<1$ of the sites. We have analyzed the degree distribution of this connected giant cluster in Fig. 3, which has scale-free behavior, and for any temperature $T<T_c$ there is the same degree exponent, $\gamma=3$, as for the original BA network. We note an interesting feature of the magnetization curves in Fig. 1 that cross each other at the transition point of the pure system, at $T_c^{(0)}=2$, having a value of $m(T_c^{(0)})=0.58$, for any strength of disorder. This property follows from the fact that for a given realization of the disorder the optimal set at $T=T_c^{(0)}$ only depends on the sign of the sum of fluctuations of given couplings [cf. some set of sites is connected (disconnected) to the giant cluster only for positive (negative) accumulated fluctuations] and does not depend on the actual value of $\Delta>0$.

We can thus conclude the following picture about the evolution of the optimal set. This is basically one large connected cluster with \mathcal{N} sites, immersed in the sea of isolated vertices. With increasing temperature more and more loosely connected sites are dissolved from the cluster, but for $T<T_c$ we have $\mathcal{N}/N=m(T)>0$ and the cluster has the same type of scale-free character as the underlying network. On the contrary, above the phase-transition point, $T_c^{(0)}+\Delta>T>T_c$, the large cluster has only a finite extent, $\mathcal{N}<\infty$. The

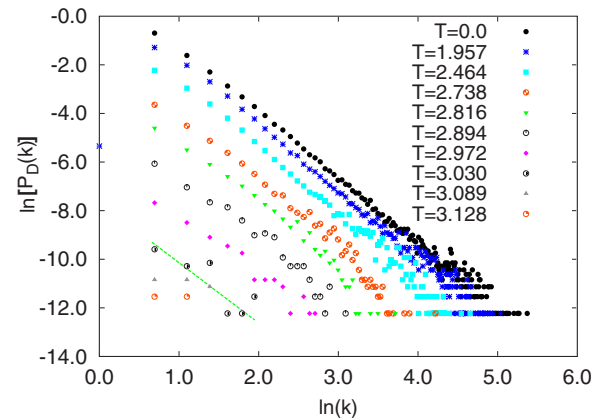


FIG. 3. (Color online) Degree distribution of the largest cluster at different temperatures in a finite network with $N=2048$. The dashed straight line indicates the range of the critical temperature.

order of the transition depends on the way \mathcal{N} behaves close to T_c . A first-order transition, i.e., phase coexistence at T_c , does not fit the above scenario. Indeed, as long as $\mathcal{N} \sim N$ the same type of continuous erosion of the large cluster should take place, i.e., the transition is of second order for any strength $\Delta > 0$ of the disorder. Approaching the critical point one expects the following singularities: $m(T) \sim (T_c - T)^\beta$ and $\mathcal{N} \sim (T - T_c)^{-\nu'}$. Finally, at $T = T_c$, the large cluster has $\mathcal{N} \sim N^{1-x}$ sites, with $x = \beta/\nu'$.

B. Distribution of the finite-size transition temperatures

The first step in the study of the critical singularities is to locate the position of the phase-transition point. In this respect it is not convenient to use the magnetization, which approaches zero very smoothly, see the inset of Fig. 1, so that there is a relatively large error by calculating T_c in this way. One might have, however, a better estimate by defining for each given sample, say α , a finite-size transition temperature $T_c(\alpha, N)$, as has been made for regular lattices [25,26,31]. For a network we use a condition for the size of the connected component, $\mathcal{N}(T) \approx AN^{1-x}$, in which x is the magnetization critical exponent and $A = O(1)$ is a free parameter, from which the scaling form of the distribution is expected to be independent. The calculation is made self-consistently: For a fixed A and a starting value of $x_s = x_1$, we have determined the distribution of the finite-size transition temperatures and at their average value we have obtained an estimate for the exponent, $x = x_2$. Then the whole procedure is repeated with $x_s = x_2$, etc., until a good convergence is obtained. Fortunately, the distribution function, $p(T_c, N)$, has only a weak x dependence; thus it was enough to make only two iterations. We have started with a logarithmic initial condition, $\mathcal{N}(T) \approx A \ln N$, which means formally $x_1 = 1$ and we have obtained $x_2 = 0.69$. Then in the next step the critical exponents are converged within the error of the calculation and they are found to be independent of the value of A , which has been set to be $A = 1, 2$, and 3 .

The distribution of the finite-size critical temperatures calculated with $x_2 = 0.69$ and $A = 3$ are shown in Fig. 4 for different sizes of the network. One can observe a shift of the position of the maxima as well as a shrinking of the width of the distribution with an increasing size of the network. The shift of the average value, $T_c^{\text{av}}(N)$, is asymptotically given by

$$T_c^{\text{av}}(N) - T_c(\infty) \sim N^{-1/\tilde{\nu}'}, \quad (5)$$

whereas the width, characterized by the mean standard deviation, $\Delta T_c(N)$, scales with another exponent, ν' , as

$$\Delta T_c(N) \sim N^{-1/\nu'}. \quad (6)$$

Using Eq. (5) from a three-point fit we have obtained $\tilde{\nu}' = 3.8(2)$ and $T_c(\infty) = 3.03(2)$. We have determined the position of the transition point in the infinite system, $T_c(\infty)$, in another way by plotting the difference $T_c^{\text{av}}(N) - T_c(\infty)$ versus N in a log-log scale for different values of $T_c(\infty)$; see Fig. 5. At the true transition point according to Eq. (5) there is an asymptotic linear dependence, which is indeed the case

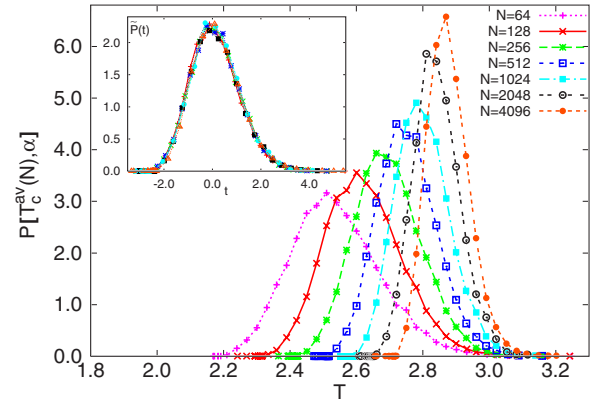


FIG. 4. (Color online) Distribution of the finite-size transition temperatures for different sizes of the BA network. Inset: Scaling collapse of the data in terms of $t = [T_c(N) - T_c^{\text{av}}(N)] / \Delta T_c(N)$, using the scaling form in Eqs. (5) and (6) with $\nu' = 3.8(2)$ and $\tilde{\nu}' = 5.6(2)$.

around $T_c(\infty) = 3.03(2)$ and the slope of the line is compatible with $1/\tilde{\nu}' = 0.27(1)$.

For the width exponent, ν' , we obtained from Eq. (6) with two-point fit the estimate $\nu' = 5.6(2)$. With these parameters the data in Fig. 4 can be collapsed to a master curve as shown in the inset of Fig. 4. This master curve does not look symmetric, at least for the finite sizes used in the present calculation, and can be well fitted by a modified Gumbel distribution, $G_\omega(-y) = \omega^\omega / \Gamma(\omega) [\exp(-y - e^{-y})]^\omega$, with a parameter $\omega = 4.2$. We note that the same type of fitting curve has already been used in Ref. [35]. For other values of the initial parameter, $A = 1$ and 2 , the estimates of the critical exponents as well as the position of the transition point are found to be stable and stand in the range indicated by the error bars.

Equations (5) and (6) are generalizations of the relations obtained in regular d -dimensional lattices [36–40] in which N is replaced by L^d , L being the linear size of the system, and therefore instead of ν' and $\tilde{\nu}'$ we have $\nu = \nu'/d$ and $\tilde{\nu}$

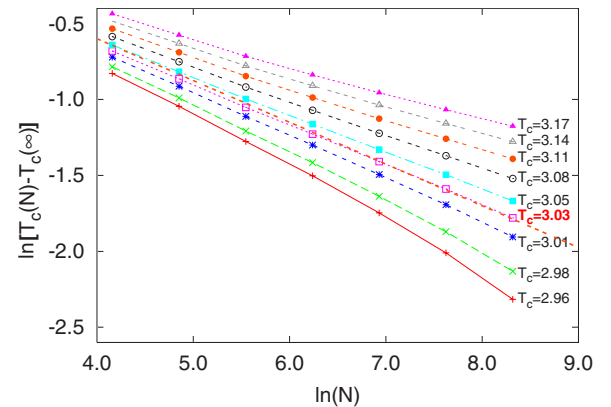


FIG. 5. (Color online) Shift of the average finite-size transition temperatures $T_c^{\text{av}}(N) - T_c(\infty)$ vs N in a log-log scale plotted for different values of $T_c(\infty)$. The lines connecting the points at the same $T_c(\infty)$ are guides for the eye. At the true transition point the asymptotic behavior is linear which is indicated by a dotted straight line.

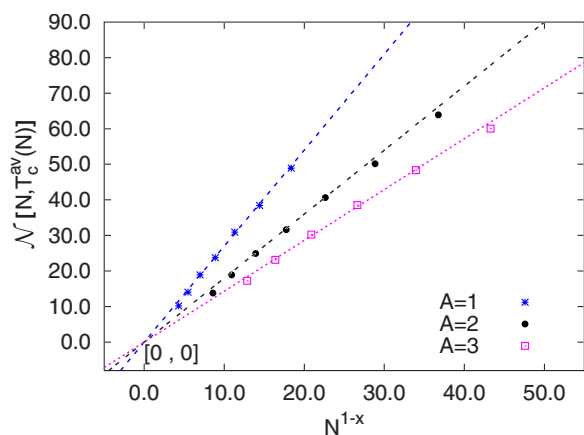


FIG. 6. (Color online) Size dependence of the critical cluster at the average finite-size critical temperature as a function of N^{1-x} with $x=0.65$. The data points for different initial parameters, A , are well described by straight lines, which are guides to the eye.

$=\tilde{\nu}'/d$, respectively. Generally at a random fixed point the two characteristic exponents are equal and satisfy the relation [41] $\nu' = \tilde{\nu}' \geq 2$. This has indeed been observed for the 2D [31] and 3D [25,26] random bond Potts models for large q at disorder induced critical points. On the other hand, if the transition stays first-order there are two distinct exponents [35,42]: $\tilde{\nu}' = 1$ and $\nu' = 2$.

Interestingly, our results on the distribution of the finite-size transition temperatures in networks are different from those found in regular lattices. Here the transition is of second order but still there are two distinct critical exponents, which are completely different from those at a disordered first-order transition. For our system $\nu' > \tilde{\nu}'$, which means that disorder fluctuations in the critical point are dominant over a deterministic shift of the transition point. A similar trend is observed about the finite-size transition parameters in the random transverse-field Ising model [43], the critical behavior of which is controlled by an infinite disorder fixed point. In this respect, the RBPM in scale-free networks can be considered a different realization of an infinite disorder fixed point.

C. Size of the critical cluster

Having the distribution of the finite-size transition temperatures we have calculated the size of the largest cluster at $T_c^{av}(N)$, which is expected to scale as $\mathcal{M}[N, T_c^{av}(N)] \sim N^{1-x}$. Then from a two-point fit we have obtained an estimate for the magnetization exponent: $x=0.66(1)$. We have also plotted $\mathcal{M}[N, T_c^{av}(N)]$ versus N^{1-x} in Fig. 6 for different initial param-

eters A . Here we have obtained an asymptotic linear dependence with an exponent, $x=0.65(1)$, which agrees with the previous value within the error of the calculation.

V. DISCUSSION IN TERMS OF OPTIMAL COOPERATION

In this paper, we have studied the properties of the Potts model for large values of q on scale-free evolving complex networks, such as the BA network, both for homogeneous and random ferromagnetic couplings. This problem is equivalent to an optimal cooperation problem in which the agents try to optimize the total sum of the benefits coming from pair cooperations (represented by the Potts couplings) and the total sum of the support which is the same for each cooperating project (given by the temperature of the Potts model). The homogeneous problem is shown exactly to have two distinct states: Either all the agents cooperate with each other or there is no cooperation between any agents. There is a strong first-order phase transition: By increasing the support the agents stop cooperating at a critical value.

In the random problem, in which the benefits are random and depend on the pairs of the cooperating agents, the structure of the optimal set depends on the value of the support. Typically the agents are of two kinds: A fraction of m belongs to a large cooperating cluster, whereas the others are isolated, representing one man's projects. With increasing support more and more agents are split off the cluster, thus its size, as well as m , is decreasing, but the cluster keeps its scale-free topology. For a critical value of the support, m goes to zero continuously and the corresponding singularity is characterized by nontrivial critical exponents. This transition, as shown by the numerically calculated critical exponents for the BA network, belongs to a new universality class. One interesting feature of it is that the distribution of the finite-size transition points is characterized by two distinct exponents and the width of the distribution is dominated over the shift of the average transition point, which is characteristic at an infinite disorder fixed point [43].

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